

# MAURER-CARTAN ELEMENTS IN THE LIE MODELS OF FINITE SIMPLICIAL COMPLEXES

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**ABSTRACT.** In a previous work, we have associated a complete differential graded Lie algebra to any finite simplicial complex in a functorial way. Similarly, we have also a realization functor from the category of complete differential graded Lie algebras to the category of simplicial sets. We have already interpreted the homology of a Lie algebra in terms of homotopy groups of its realization. In this paper, we begin a dictionary between models and simplicial complexes by establishing a correspondence between the Deligne groupoid of the model and the connected components of the finite simplicial complex.

Let  $\mathrm{MC}(L)$  be the set of Maurer-Cartan elements of a differential graded Lie algebra  $(L, d)$  over  $\mathbb{Q}$  (henceforth DGL). The group  $L_0$  of elements of degree 0, endowed with the Baker-Campbell-Hausdorff product, acts on  $\mathrm{MC}(L)$  by

$$x \mathcal{G} z = e^{\mathrm{ad}_x}(z) - \frac{e^{\mathrm{ad}_x} - 1}{\mathrm{ad}_x}(dx),$$

with  $x \in L_0$  and  $z \in \mathrm{MC}(L)$ . We denote by  $\widetilde{\mathrm{MC}}(L)$  the orbit space for this action.

In [1], we construct a functor  $\mathcal{L}$  from the category of finite simplicial complexes to the category of complete differential graded Lie algebras (henceforth cDGL),  $X \mapsto \mathcal{L}_X$ . Rational homotopy has been mainly introduced and used for simply connected spaces ([5], [10], [11]). In [11], there is also an extension to non-simply connected spaces over  $\mathbb{R}$  via fiber bundles (see [7] for an adaptation to  $\mathbb{Q}$ ). Recently, the classical approach has been extended to non-simply connected spaces in [6] and the functor  $\mathcal{L}$  gives the corresponding extension for DGL's.

In this paper we prove the following relation between  $\mathcal{L}_X$  and the topology of  $X$ .

**Theorem.** *For any finite simplicial complex  $X$  there is a bijection*

$$\pi_0(X_+) \cong \widetilde{\mathrm{MC}}(\mathcal{L}_X),$$

where  $X_+ = X \sqcup \{*\}$ .

The case of the interval  $X = [0, 1]$  was solved in [2]. In Section 1, we make the necessary recalls on Maurer-Cartan elements and the functor  $\mathcal{L}$ . Section 2 is devoted to a decomposition of  $\mathcal{L}_X$  when  $X$  is connected. Finally, the proof of the Theorem is done in Section 3.

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1. FUNCTOR  $\mathcal{L}$  AND MAURER-CARTAN ELEMENTS

Recall that a DGL  $(L, d)$  is *complete* if  $L = \varprojlim_n L/L^{[n]}$  where  $L^{[n]}$  denotes the sequence of ideals defined by

$$L^{[1]} = L, \quad \text{and } L^{[n+1]} = [L, L^{[n]}], \quad n \geq 2.$$

When  $V$  is finite dimensional,  $\widehat{\mathbb{L}}(V) = \varprojlim_n \mathbb{L}(V)/\mathbb{L}(V)^{[n]}$  is the completion of the free graded Lie algebra  $\mathbb{L}(V)$ .

Let  $(L, d)$  be a cDGL. An element  $u \in L_{-1}$  is a *Maurer-Cartan element* if

$$du = -\frac{1}{2}[u, u].$$

In [8], R. Lawrence and D. Sullivan construct a cDGL  $\mathcal{L}_I$  that is, in a sense that we will precise later, a model for the interval  $I = [0, 1]$ . More precisely,

$$\mathcal{L}_I = (\widehat{\mathbb{L}}(a, b, x), d),$$

where  $a$  and  $b$  are Maurer-Cartan elements and  $x$  is an element of degree 0 with

$$dx = \text{ad}_x b + \frac{\text{ad}_x}{e^{\text{ad}_x} - 1}(b - a) = [x, b] + \sum_{n=0}^{\infty} \frac{B_n}{n!} \text{ad}_x^n(b - a).$$

Here the  $B_n$  are the well known Bernoulli numbers. This model has been described in detail in [9], [4].

In a cDGL  $(L, d)$ , two Maurer-Cartan elements  $u_1$  and  $u_2$  are *equivalent* if they are in the same orbit for the gauge action. By construction, this is equivalent to the existence of a morphism of DGL's,

$$f: \mathcal{L}_I \rightarrow (L, d)$$

with  $f(a) = u_1$  and  $f(b) = u_2$ . The map  $f$  is called *a path from  $u_1$  to  $u_2$* . The set of equivalence classes of Maurer-Cartan elements is denoted  $\widetilde{MC}(L)$ .

Our purpose is the determination of  $\widetilde{MC}(L)$  for a family of cDGL's directly related to topology. In fact the cDGL  $\mathcal{L}_I$  is the first example of a Lie model for a general simplicial complex. More generally, there is a functor  $\mathcal{L}$ , unique up to isomorphism,  $X \mapsto \mathcal{L}_X$ , from the category of finite simplicial complexes to the category of cDGL's. As any finite simplicial complex is a subcomplex of some  $\Delta^n$ , it is sufficient to construct the models,  $\mathcal{L}_{\Delta^n}$ , of the  $\Delta^n$ 's.

**Proposition 1.** [1, Theorem 2.8] *The cDGL  $\mathcal{L}_{\Delta^n}$  is defined, up to isomorphism, by the following properties.*

- (i) *The cDGL's  $\mathcal{L}_{\Delta^n}$  are natural with respect to the injections of the subcomplexes  $\Delta^p$ , for all  $p < n$ .*
- (ii) *For  $n = 0$ , we have  $\mathcal{L}_{\Delta^0} = (\widehat{\mathbb{L}}(a), d)$  where  $a$  is a Maurer-Cartan element.*
- (iii) *The linear part  $d_1$  of the differential of  $\mathcal{L}_{\Delta^n}$  is the desuspension of the differential  $\delta$  of the chain complex  $C_*(\Delta^n)$ .*

In the case  $\Delta^1 = [0, 1]$ , we recover the Lawrence-Sullivan construction. For each finite simplicial complex,  $X$ , contained in  $\Delta^n$ , the Lie subalgebra  $\widehat{\mathbb{L}}(s^{-1}C_*(X))$  is preserved by the differential of  $\mathcal{L}_{\Delta^n}$  and gives a *model*  $\mathcal{L}_X$  of  $X$ .

When  $a$  is a Maurer-Cartan element in  $\mathcal{L}_X$ , we denote by  $d_a$  the perturbed differential  $d_a = d + \text{ad}_a$ . The first properties of  $\mathcal{L}_X = (\widehat{\mathbb{L}}(W), d)$  are contained in the following statements extracted from [1] and [3].

- (i) If  $d_1$  denotes the linear part of the differential  $d$ , then  $(W, d_1)$  is isomorphic to the desuspension of the simplicial chain complex  $C_*(X)$  of  $X$ .
- (ii) If  $f: X \rightarrow Y$  is the inclusion of a subcomplex, then  $\mathcal{L}_f: \mathcal{L}_X \rightarrow \mathcal{L}_Y$  is equal to  $\widehat{\mathbb{L}}(s^{-1}C_*(f))$ .
- (iii)  $H(\mathcal{L}_X) = 0$  ([3], Theorem 4.1).
- (iv) If  $X$  is simply connected, and  $a$  is the Maurer-Cartan element associated to a 0-simplex, then  $(\widehat{\mathbb{L}}(W), d_a)$  is quasi-isomorphic to the usual rational Quillen model of  $X$  ([1], Theorem 7.4(ii)).
- (v) If  $X$  is connected and  $a$  is the Maurer-Cartan element associated to a 0-simplex, then  $H_0(\widehat{\mathbb{L}}(W), d_a)$  is isomorphic to the Malcev Completion of  $\pi_1(X)$  ([1], Theorem 9.1).

Recall that the Lawrence-Sullivan interval  $\mathcal{L}_I$  is isomorphic to the cylinder construction ([12]) on a Maurer-Cartan element ([3, Theorem 6.3]). More precisely, consider the cDGL  $(\widehat{\mathbb{L}}(a, c, y), d)$  with  $|y| = 0$ ,  $|c| = -1$ ,  $da = -\frac{1}{2}[a, a]$ ,  $dy = c$  and  $dc = 0$  that we equip with a derivation  $s$  of degree  $+1$ , defined by  $s(a) = y$ ,  $s(c) = s(y) = 0$ . Then the morphism

$$\psi: (\widehat{\mathbb{L}}(a, b, x), d) \rightarrow (\widehat{\mathbb{L}}(a, c, y), d) \quad (1)$$

defined by  $\psi(a) = a$ ,  $\psi(b) = e^{sd+ds}(a)$ ,  $\psi(x) = y$  is an isomorphism of DGL's. In particular,

$$\psi(b) = a + c + \sum_{n \geq 1} \frac{(sd)^n}{n!}(a) = e^{\text{ad}_{-y}}(a) + \frac{e^{\text{ad}_{-y}} - 1}{\text{ad}_{-y}}(c).$$

**Definition 2.** Two Maurer-Cartan elements  $u, v$  in a cDGL  $(\widehat{\mathbb{L}}(V), d)$  are called *equivalent of order  $r$*  if there is a morphism

$$\varphi: (\widehat{\mathbb{L}}(a, b, x), d) \rightarrow (\widehat{\mathbb{L}}(V), d)$$

with  $\varphi(x) \in \mathbb{L}^{\geq r}(V)$ ,  $\varphi(a) = u$  and  $\varphi(b) = v$ . We denote this relation by  $u \sim_{O(r)} v$ .

This relation is a key-point in the proof of Proposition 5. We end this section with two properties of  $\sim_{O(r)}$ .

**Lemma 3.** Let  $u$  be a Maurer-Cartan element in  $(\widehat{\mathbb{L}}(V), d)$ . We suppose  $u = v + w$  with  $w \in \mathbb{L}^{\geq r}(V)$ , and the existence of an element  $z \in \mathbb{L}^{\geq r}(V)$  with  $dz = w + t$  and  $t \in \mathbb{L}^{\geq r+1}(V)$ . Then, we have  $u \sim_{O(r)} v + w'$  with  $w' \in \mathbb{L}^{\geq r+1}(V)$ .

*Proof.* Let  $f: (\widehat{\mathbb{L}}(a, c, y), d) \rightarrow (\widehat{\mathbb{L}}(V), d)$  be the morphism defined by  $f(a) = u$ ,  $f(y) = -z$  and  $f(c) = -dz$ . Then  $f \circ \psi$  is a path in  $(\widehat{\mathbb{L}}(V), d)$  with  $f\psi(a) = u$ ,  $f\psi(x) = -z$ . To determine  $f\psi(b)$ , we first observe that

$$\psi(b) = a + c + \sum_{n \geq 1} \frac{(sd)^n}{n!}(a).$$

Remark also that  $f(sd)^n(a) \in \mathbb{L}^{\geq r+1}(V)$ , for  $n \geq 1$ . Therefore

$$f \circ \psi(b) \in f(a) + f(c) + \mathbb{L}^{\geq r+1}(V) = u - dz + \mathbb{L}^{\geq r+1}(V) = v - t + \mathbb{L}^{\geq r+1}(V),$$

with  $t \in \mathbb{L}^{\geq r+1}(V)$ .  $\square$

**Lemma 4.** *Let  $(u_r)_{r \geq n_0}$  be a sequence of Maurer-Cartan elements in  $(\widehat{\mathbb{L}}(V), d)$  such that  $u_r = z + v_r$  with  $v_r \in \mathbb{L}^{\geq r}(V)$ . If  $u_r \sim_{O(r)} u_{r+1}$  for each  $r \geq n_0$ , then we have  $u_{n_0} \sim_{O(n_0)} z$ .*

*Proof.* By hypothesis, for  $r \geq n_0$  there is a morphism

$$\varphi_r: (\widehat{\mathbb{L}}(a, b, x), d) \rightarrow (\widehat{\mathbb{L}}(V), d)$$

with  $\varphi_r(a) = u_r$ ,  $\varphi_r(b) = u_{r+1}$  and  $\varphi_r(x) \in \mathbb{L}^{\geq r}(V)$ . For  $r > n_0$ , we define  $w_r$  to be the Baker-Campbell-Hausdorff product

$$w_r = \varphi_{n_0}(x) * \varphi_{n_0+1}(x) * \cdots * \varphi_{r-1}(x).$$

From the associativity established in [8], the element  $w_r$  is a path from  $u_{n_0}$  to  $u_r$ . We form the infinite product

$$w = \varphi_{n_0}(x) * \varphi_{n_0+1}(x) * \cdots$$

which is well defined in  $\widehat{\mathbb{L}}(V)$  as the limit of the  $w_r$ . Now we claim that the element  $w$  is a path of order  $n_0$  from  $u_{n_0}$  to  $z$ ; i.e., we have  $u_{n_0} \sim_{O(n_0)} z$ . Consider the element

$$y = dw - [w, z] - \sum_{n \geq 0} \frac{B_n}{n!} \text{ad}_w^n(z - u_{n_0}),$$

where the  $B_n$  are the Bernoulli numbers. The element  $y$  has the same image in  $\mathbb{L}(V)/\mathbb{L}^{\geq r}(V)$  than

$$dw_r - [w_r, u_r] - \sum_{n \geq 0} \frac{B_n}{n!} \text{ad}_{w_r}^n(u_r - u_{n_0}).$$

This last expression is equal to 0 because  $w_r$  is a path from  $u_{n_0}$  to  $u_r$ . This implies  $y = 0$  and proves the result.  $\square$

## 2. MODEL OF A FINITE CONNECTED SIMPLICIAL COMPLEX

**Proposition 5.** *Let  $X$  be a connected finite simplicial complex of dimension  $n$ , then we have an isomorphism of cDGL's*

$$\mathcal{L}_X \cong (\widehat{\mathbb{L}}(V), d) \widehat{\Pi}_i (\widehat{\mathbb{L}}(u_i, v_i), d)$$

where  $dv_i = u_i$ ,  $du_i = 0$ ,  $V = V_{\leq n-1}$ ,  $V = \mathbb{Q}a \oplus V_{\geq 0}$ ,  $a$  is a Maurer-Cartan element and  $\widehat{\Pi}$  denotes the completion of the coproduct. Moreover, the differential of any  $x \in V_{\geq 0}$  verifies

$$dx + [a, x] \in \widehat{\mathbb{L}}^{\geq 2}(V_{\geq 0}).$$

*Proof.* By Lemma 6, this is true if  $\dim X = 1$ . Proceed by induction on  $n$ . We can therefore suppose that  $X = Y \cup \bigcup_{j=1}^k \Delta_j^n$  and

$$(\mathcal{L}_Y, d) \cong (\widehat{\mathbb{L}}(V), d) \widehat{\Pi}_i (\widehat{\mathbb{L}}(u_i, v_i), d)$$

with  $n \geq 2$ ,  $\dim Y \leq n - 1$ ,  $V = V_{\leq n-2} = \mathbb{Q}a \oplus W$ ,  $W = W_{\geq 0}$ ,  $|v_i| \leq n - 2$ ,  $dv_i = u_i$ . We set  $u'_i = u_i + [a, v_i]$  and we get an isomorphism of DGL's

$$(\widehat{\mathbb{L}}(V), d_a) \widehat{\Pi} \widehat{\Pi}_i (\widehat{\mathbb{L}}(u'_i, v_i), d_a) \rightarrow (\mathcal{L}_Y, d_a),$$

with  $d_a v_i = u'_i$ ,  $d_a u'_i = 0$ . Now, by construction of the model  $\mathcal{L}_X$ , there are cycles  $\Omega_j \in (\mathcal{L}_Y)_{n-2}$  such that

$$(\mathcal{L}_X, d_a) = (\mathcal{L}_Y \widehat{\Pi} \widehat{\Pi}_{j=1}^k \mathbb{L}(x_j), d_a), \quad |x_j| = n - 1, \quad d_a x_j = \Omega_j.$$

Since the inclusion  $(\widehat{\mathbb{L}}(V), d_a) \hookrightarrow (\widehat{\mathbb{L}}(V), d_a) \widehat{\Pi} \widehat{\Pi}_i (\widehat{\mathbb{L}}(u'_i, v_i), d_a)$  is a quasi-isomorphism, we can choose  $\Omega_j \in \widehat{\mathbb{L}}(W)$ .

Let  $(x_j)_{j \in \mathcal{A}}$  the family of the  $x_j$ 's such that the differential  $dx_j = \Omega_j$  has a non-zero linear part  $\Omega_j^1$ . We set  $\mathcal{B} = \{1, \dots, k\} \setminus \mathcal{A}$  and denote by  $\mathcal{K}$  the ideal generated by  $\{x_j, \Omega_j^1 \mid j \in \mathcal{A}\}$ . If  $V'$  is a direct summand of  $\oplus_{j \in \mathcal{A}} \mathbb{Q}\Omega_j^1$  in  $V$ , we have an isomorphism  $(\widehat{\mathbb{L}}(V'), d) \cong (\widehat{\mathbb{L}}(V), d)/\mathcal{K}$ . From [1, Proposition 2.4], we deduce that the canonical surjection  $\rho: (\widehat{\mathbb{L}}(V), d) \rightarrow (\widehat{\mathbb{L}}(V), d)/\mathcal{K}$  is a quasi-isomorphism. Since the DGL  $(\widehat{\mathbb{L}}(V'), d)$  is cofibrant ([3, Proposition 5.4]), we may lift  $\rho$  in a quasi-isomorphism

$$\varphi: (\widehat{\mathbb{L}}(V'), d) \widehat{\Pi} \widehat{\Pi}_{j \in \mathcal{A}} \widehat{\mathbb{L}}(x_j, \Omega_j) \rightarrow (\widehat{\mathbb{L}}(V), d)$$

and get an isomorphism

$$\mathcal{L}_X \cong \widehat{\mathbb{L}}(V' \oplus \oplus_{j \in \mathcal{B}} \mathbb{Q}x_j) \widehat{\Pi} (\widehat{\Pi}_{j \in \mathcal{A}} \widehat{\mathbb{L}}(x_j, \Omega_j) \widehat{\Pi}_i \widehat{\mathbb{L}}(u_i, v_i)).$$

□

**Lemma 6.** *Let  $X$  be a 1-dimensional connected finite simplicial complex, then we have an isomorphism of cDGL's*

$$\mathcal{L}_X \cong (\widehat{\mathbb{L}}(V), d) \widehat{\Pi} (\widehat{\mathbb{L}}(u_i, v_i), dv_i = u_i),$$

with  $V = \mathbb{Q}a \oplus V_0$ ,  $da = -\frac{1}{2}[a, a]$  and  $dx = -[a, x]$  for any  $x \in V_0$ .

*Proof.* Let  $x_0$  be a vertex of  $X$  and let  $a$  denote the corresponding Maurer-Cartan element in  $\mathcal{L}_X$ . By hypothesis  $X$  is a connected finite graph, and we denote by  $\mathcal{T}$  a maximal tree in  $X$ . For each vertex  $v_i$  different from  $x_0$ , there is a unique path  $\mathcal{P}_{v_i} \in \mathcal{T}$  of minimal length from  $x_0$  to  $v_i$ . We remark that each edge in  $\mathcal{T}$  is the terminal edge of some path  $\mathcal{P}_{v_i}$  for some vertex  $v_i$  different from  $x_0$ . The vertices  $v_i$  correspond to Maurer-Cartan elements  $a_i$  in  $\mathcal{L}_X$ . To each path  $\mathcal{P}_{v_i}$  we associate the Baker-Campbell-Hausdorff product  $p_i$  of the edges composing this path.

If  $b_k$  is an edge which does not belong to  $\mathcal{T}$ , we denote by  $v_{k_0}$  and  $v_{k_1}$  its endpoints. If each of them is different from  $x_0$ , we form the loop consisting of the path  $\mathcal{P}_{v_{k_0}}$  followed by  $b_k$  and  $(\mathcal{P}_{v_{k_1}})^{-1}$ . If  $v_{k_0} = x_0$ , we form the loop consisting of  $b_k$  and  $(\mathcal{P}_{v_{k_1}})^{-1}$  and do similarly if  $v_{k_1} = x_0$ . We denote then by  $c_k$  the Baker-Campbell-Hausdorff product of the edges composing this loop.

From these two constructions, we get a morphism of DGL's

$$f: (\mathcal{L}', d) := (\widehat{\mathbb{L}}(a, a_i, p_i, c_k), d) \rightarrow \mathcal{L}_X.$$

The map  $f$  induces an isomorphism on the indecomposable elements and thus it is an isomorphism. In  $(\mathcal{L}', d)$ , for each  $i$ ,  $(\widehat{\mathbb{L}}(a, a_i, p_i), d)$  is a Lawrence-Sullivan interval

connecting  $a$  to  $a_i$ . On the other hand (see [1, Proposition 2.7]), for each  $k$  we have  $dc_k = -[a, c_k]$ .

Recall now from (1) that for each  $i$ , there is an isomorphism

$$\psi_i: (\widehat{\mathbb{L}}(a, a_i, p_i), d) \rightarrow (\widehat{\mathbb{L}}(a, u_i, v_i), d)$$

with  $\psi(a) = a$ ,  $\psi(p_i) = v_i$ ,  $du_i = 0$  and  $dv_i = u_i$ . The morphisms  $\psi_i$  can be pasted together and give an isomorphism

$$\psi: (\mathcal{L}', d) \rightarrow (\widehat{\mathbb{L}}(a, u_i, v_i, c_k), d)$$

with  $dc_k = -[a, c_k]$  and  $dv_i = u_i$ . Therefore

$$\mathcal{L}_X \cong (\widehat{\mathbb{L}}(V), d) \widehat{\Pi} (\widehat{\mathbb{L}}(u_i, v_i), d)$$

with  $V = \mathbb{Q}a \oplus V_0$  and  $dx = -[a, x]$  for any  $x \in V_0$ . □

**Corollary 7.** *With the notations of Proposition 5, we have*

$$\widetilde{MC}(\mathcal{L}_X) = \widetilde{MC}(\widehat{\mathbb{L}}(V), d).$$

*Proof.* This follows directly from [3, Proposition 2.4]. □

### 3. MAURER CARTAN ELEMENTS AND CONNECTED COMPONENTS

*Proof of the Theorem.* Let  $X$  be a finite simplicial complex and denote by  $X_i$  its connected components for  $i = 1, \dots, k$ . Then

$$\mathcal{L}_X = \widehat{\Pi}_{i=1}^k \mathcal{L}_{X_i}.$$

For each  $i = 1, \dots, k$  we have

$$\mathcal{L}_{X_i} \cong (\widehat{\mathbb{L}}(V(i), d) \widehat{\Pi} (\widehat{\mathbb{L}}(u_{ij}, v_{ij}), d),$$

with  $d(u_{ij}) = v_{ij}$  and  $V(i) = \mathbb{Q}a_i \oplus V(i)_{\geq 0}$  verifies the properties established in Proposition 5. Moreover, we deduce from Corollary 7

$$\widetilde{MC}(\mathcal{L}_X) = \widetilde{MC}(\widehat{\Pi}_{i=1}^k (\widehat{\mathbb{L}}(V(i)), d)).$$

A Maurer-Cartan element  $u \in \mathcal{L}_X$  can be written in the form

$$u = \sum_{i=1}^k \lambda_i a_i + \mu,$$

where  $\mu$  is a decomposable element and  $\lambda_i \in \mathbb{Q}$ . From a short computation, we observe that all the numbers  $\lambda_i$ , except at most one, are equal to zero.

- If  $\lambda_1 \neq 0$ , then  $\lambda_1 = 1$  and we set  $a = a_1$ ,  $V = V(1)$  and  $W = \oplus_{i \geq 2} V(i)$ . We denote by  $E_r$  the subvector space of  $\mathcal{L}_X$  generated by the Lie words containing exactly  $r$  elements of  $V_{\geq 0}$ . The differential  $d$  can be written as a series  $d = \sum_{i \geq 1} d_i$ , with  $d_i(V) \subset E_i$ . By hypothesis, we have  $d_1(v) = -[a, v]$  if  $v \in V_{\geq 0}$  and  $d_1(w) = 0$  if  $w \in W$ . Remark now that since  $a$  is in degree  $-1$  and  $V \oplus W$  is finite dimensional, the ideal  $E_{\geq 1}$  generated by  $V_{\geq 0}$  is the free complete DGL on the elements  $a^r \boxtimes v_k := \text{ad}_a^r(v_k)$  and

$a^r \boxtimes w_k := \text{ad}_a^r(w_k)$ , where  $r \geq 0$ , the  $v_k$ 's run over a graded basis of  $V_{\geq 0}$  and the  $w_k$  over a graded basis of  $W$ . Recall  $v \in V_{\geq 0}$  and  $w \in W$ . A simple computation gives

$$d_1(a^r \boxtimes v) = \begin{cases} -a^{r+1} \boxtimes v, & \text{if } r \text{ is even,} \\ 0, & \text{if } r \text{ is odd,} \end{cases}$$

$$d_1(a^r \boxtimes w) = \begin{cases} 0, & \text{if } r \text{ is even,} \\ -a^{r+1} \boxtimes w, & \text{if } r \text{ is odd.} \end{cases}$$

The derivation defined by  $\theta = -\text{ad}_a - d_1$  verifies

$$\theta(a^r \boxtimes v) = \begin{cases} 0, & \text{if } r \text{ is even,} \\ -a^{r+1} \boxtimes v, & \text{if } r \text{ is odd,} \end{cases}$$

$$\theta(a^r \boxtimes w) = \begin{cases} -a^{r+1} \boxtimes w, & \text{if } r \text{ is even,} \\ 0, & \text{if } r \text{ is odd.} \end{cases}$$

Clearly we have  $\theta^2 = 0$  and  $H(E_{\geq 1}, \theta) = \widehat{\mathbb{L}}(V)$ . In particular

$$H_{-1}(E_{\geq 1}, \theta) = 0.$$

We construct a sequence of Maurer-Cartan elements  $(u_n)$  such that  $u_1 = u$ ,  $u_n - a \in E_{\geq n}$  and  $u_n \sim_{O(n)} u_{n+1}$ . Suppose  $u_n$  has been constructed, then we can write it as

$$u_n = a + \omega_n + \gamma, \quad \text{with } \omega_n \in E_n, \quad \gamma \in E_{>n}.$$

Since  $u_n$  is a Maurer-Cartan element, we have  $d_1(\omega_n) = -[a, \omega_n]$  and  $\theta(\omega_n) = 0$ . From  $H_{-1}(E_{\geq 1}, \theta) = 0$ , we deduce the existence of  $t \in E_n$  such that  $\omega_n = \theta(t)$ . This implies

$$\omega_n = -[a, t] - d_1(t).$$

Recall from (1) the morphism

$$\psi: (\widehat{\mathbb{L}}(a, b, x), d) \rightarrow (\widehat{\mathbb{L}}(a, e, c), d)$$

and construct a morphism  $\mu: (\widehat{\mathbb{L}}(a, e, c), d) \rightarrow (\widehat{\mathbb{L}}(\mathbb{Q}a \oplus V), d)$ , by  $\mu(a) = u_n$ ,  $\mu(e) = t$  and  $\mu(c) = dt$ . A short computation gives

$$\mu \circ \psi(b) = a + \gamma', \quad \gamma' \in E_{>n}.$$

The path  $\mu \circ \psi$  defines  $u_{n+1}$  such that  $u_n \sim_{O(n)} u_{n+1}$  and the result follows from Lemma 4.

• Suppose now  $\lambda_i = 0$  for  $i = 1, \dots, k$ . We write  $u = \sum_{i \geq 1} \omega_i$  with  $\omega_i \in E_i$ . Since  $u$  is a Maurer-Cartan element, we have  $d\omega_1 = 0$ . From  $H(\mathcal{L}_X, d) = 0$ , we deduce the existence of  $\omega'_1$  such that  $\omega_1 = d\omega'_1$  and Lemma 3 implies  $u \sim_{O(1)} u_2$  with  $u_2 \in E_{\geq 2}$ . With the same process, we get a sequence of Maurer-Cartan elements  $u_n \in E_{\geq n}$  such that  $u_n \sim_{O(n)} u_{n+1}$ . Finally Lemma 4 gives  $u \sim 0$ .  $\square$

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